Note: (elementary matrices and row reduction)

Any operation used in row-reducing a matrix can be expressed as left multiplication by an invertible matrix.

$$A = \begin{bmatrix} 3 & -8 & 6 \\ -1 & 7 & -2 \end{bmatrix}$$

Swap the rows: left multiply

by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -8 & 6 \\ -1 & 7 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 7 & -2 \\ 3 & -8 & 6 \end{bmatrix}$$

Then we multiply R1 by 3 and add it to R2:

Left multiplication by

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}:$$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 7 & -2 \\ 3 & -8 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 7 & -2 \\ 0 & 13 & 0 \end{bmatrix}$$

The general form of row operations on $A \in M_{n\times m}(\mathbb{C})$ are given by the following matrices in $M_n(\mathbb{C})$:

1) Multiply row i by a scalar α:

matrix B with

$$B_{s,t} = \begin{cases} 0, & s \neq t \\ \alpha, & s = t = i \\ 1, & s = t \neq i \end{cases}$$

2) Swap row i with row j!

Matrix B with

$$B_{s,t} = \begin{cases} 1, & s=i, t=j \\ 1, & s=j, t=i \\ 1, & s=t \neq i \text{ or } j \\ 0, & \text{otherwise} \end{cases}$$

This matrix is its own inverse.

3) Add α times row i to row j:

Matrix B

$$B_{s,t} = \begin{cases} 1, & s = t \\ \alpha, & s = j,\ t = i \\ 0, & \text{otherwise} \end{cases}$$

Observe: The matrices produced in 3) are always invertible; the matrices produced in 1) are invertible if $\alpha \neq 0$.

Proposition: Consider the equation $Ax = b$ for $b \in \mathbb{C}^n$ and $A \in M_{n \times m}(\mathbb{C})$.

Then

1) a solution x is unique iff the echelon form of A has a pivot in every column.

(i.e. there are no columns with all zero entries)

2) $\exists$ a solution for all b iff the echelon form of A has a pivot in every row.

(i.e. there are no rows with all entries zero)

3) $\exists$ a unique solution iff the echelon form of A has a pivot in every row & column.

proof: 1) Observe that if the i^{th} column of the echelon form of A is entirely zero, then the i^{th} entry of x can be any number $\Rightarrow$ non-uniqueness of solutions.

The reverse implication follows using the same argument.

2) Suppose the echelon form of A has a row of all zeros. WLOG, we may assume it is the last row of A. Then let A_0 be the echelon form of A and let

$$b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

i.e.

$$b \in \mathbb{C}^n, \quad b_i = \begin{cases} 0, & i \neq n \\ 1, & i = n \end{cases}$$

Then no matter the vector $x \in \mathbb{C}^m$, $(A_0 x)_n = 0$ (the n^{th} coordinate of x is always zero). So

$$A_0 x = b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

has no solution x.

Then let E be the product, in order, of all elementary row matrices applied to A, so

$$EA = A_0.$$

Then E is invertible since all the elementary row matrices are invertible.

Then writing

$$EAx = A_0 x = b$$ and

apply E^{-1} on the left,

we obtain

$$Ax = E^{-1} b$$ has

no solution.

Conversely, if there are no rows that are entirely zero, we can solve for a solution via back substitution.

3) Trivial - combine 1) and 2)

$\square$

Note: (non-uniqueness in 1))

Suppose $Ax = b$ has two non-unique solutions x_0 and x_1. Then if $Ax_2 = b_0$ for some x_2,

then

$$A(x_2 + \overbrace{x_0 - x_1}^{\neq 0 \text{ since } x_0 \neq x_1})$$

$$= Ax_2 + Ax_0 - Ax_1 \text{ by linearity}$$

$$= b_0 + b - b = b_0.$$

So with

$$x_3 = x_2 + x_0 - x_1 \mp x_2,$$

$$Ax_3 = b_0 = Ax_2.$$

So all equations have non-unique solutions if even one does!

Corollary: Let $v_1, \dots, v_m \in \mathbb{C}^n$

and $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} \in M_{n\times m}(\mathbb{C})$.

Then

1) $\{v_1, \dots, v_m\}$ is linearly independent iff the echelon form of A has a pivot in every column.

2) $\{v_1, \ldots, v_m\}$ is spanning iff the echelon form of A has a pivot in every row.

3) $\{v_1, \ldots, v_m\}$ is a basis iff the echelon form of A has a pivot in every row & column.

proof: For $\alpha_1, \cdots, \alpha_m \in \mathbb{C}$ and $b \in \mathbb{C}^n$, we may write

$$\sum_{i=1}^{m} \alpha_i v_i = b \text{ as}$$

$$A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = b \text{ where}$$

$A \in M_{n \times m}(\mathbb{C})$ is the matrix formed using $\{v_1, \ldots, v_m\}$ as the columns.

Specifically, writing each v_i, $1 \le i \le m$, as a column vector in terms of the standard basis,

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}.$$

1) $\{v_1, \ldots, v_m\}$ linearly independent $\Longleftrightarrow$

$$\sum_{i=1}^{m} \alpha_i v_i = 0 \Rightarrow \alpha_i = 0 \quad \forall\, 1 \le i \le m$$

$$\Leftrightarrow \quad A\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = 0$$

$$\Rightarrow \quad \alpha_i = 0 \ \forall \ 1 \le i \le m$$

$\Leftrightarrow$ the zero vector is the unique solution to

$$Ax = 0$$

$\Leftrightarrow$ A has a pivot in every column, by the previous proposition.

2) $\{v_1, \dots, v_m\}$ spanning

$\iff \forall\, b \in \mathbb{C}^n,\ \exists\, \alpha_1, \dots, \alpha_m \in \mathbb{C},$

$$\sum_{i=1}^{m} \alpha_i v_i = b$$

$\iff \forall\, b \in \mathbb{C}^n,\ \exists\, \alpha_1, \dots, \alpha_m \in \mathbb{C}$

$$A \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = b$$

$\iff$ the equation $Ax = b$ has a solution $\forall\, b \in \mathbb{C}^n$

$\Longleftrightarrow$ A has a pivot in every row, by the previous proposition

3) Trivial, combine 1) and 2)

$\square$

Lemma: Let $T: V \to W$ be an isomorphism. Then

1) $S \subseteq V$ is linearly independent iff $T(S)$ is linearly independent

2) $S \subseteq V$ is spanning iff $T(S)$ is spanning.

Proof: 1) $\Rightarrow$ 2) Suppose S is linearly dependent and let $x_1, \ldots, x_n \in S$, $n \in \mathbb{N}$.

Suppose $$\sum_{i=1}^{n} \alpha_i T(x_i) = 0_W$$

for some scalars $\alpha_1, \ldots, \alpha_n$.

Then by linearity,

$$T\left(\sum_{i=1}^{n} \alpha_i x_i\right) = 0_W$$

Since T is an isomorphism,

$\ker(T) = \{0_V\} \Rightarrow$

$$\sum_{i=1}^{n} \alpha_i x_i = 0_V$$

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$

by linear independence of S.

$\Leftarrow$) is identical when replacing S with $T(S)$ and T with T^{-1}.

2) $\Rightarrow$) Let $y \in W = T(V)$

since T is an isomorphism.

Then $\exists\, x \in V$,

$T(x) = y$. Since S

is spanning for V,

$\exists\, n \in \mathbb{N}$, $x_1, \ldots, x_n \in S$, and

$\alpha_1, \ldots, \alpha_n$ scalars with

$$x = \sum_{i=1}^{n} \alpha_i x_i .$$

Then

$$y = T(x) = T\left(\sum_{i=1}^{n} \alpha_i x_i\right)$$

$$= \sum_{i=1}^{n} \alpha_i T(x_i)$$

$$\in \text{span}(T(S))$$

by linearity of T.

Therefore, $\text{span}(T(S)) = W$

$(\Leftarrow)$ is identical when replacing S with $T(S)$ and T with T^{-1}. $\square$

Theorem: Let V be a finite-dimensional vector space. Then $\dim(V)$ is well-defined.

proof: Assume $V = \mathbb{C}^n$ for some $n \in \mathbb{N}$.

We know $\{e_1, \dots, e_n\}$ is a basis for $\mathbb{C}^n$.

Suppose $\{v_1, \dots, v_m\}$ is a basis for $\mathbb{C}^n$, $m \in \mathbb{N}$.

Writing each v_i, $1 \le i \le m$, in terms of the standard basis, we can construct the matrix

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} \in M_{n \times m}(\mathbb{C}).$$

By the previous corollary, the echelon form of A has a pivot in every row (n pivots) and a pivot in every column (m pivots).

Then $n = m$ since the number of pivots is the same whether you consider rows or columns.

Now let V be an arbitrary finite-dimensional complex vector space. Then V is isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$. Let $T: V \to \mathbb{C}^n$ be an isomorphism.

Then by the previous lemma, $\{T^{-1}(e_i)\}_{i=1}^{n}$ is a basis for V.

If $\{w_1, \ldots, w_m\}$ is another basis for V, then again by the lemma,

$\{T(w_j)\}_{j=1}^{m}$ is a basis for $\mathbb{C}^n$

$\Rightarrow$ $m = n$.

$\square$